FRACTIONAL HARDY–SOBOLEV–MAZ'YA INEQUALITY ON BALLS AND HALFSPACES

BARTŁOMIEJ DYDA

ABSTRACT. We prove fractional Hardy–Sobolev–Maz'ya inequality for balls and a half-space, partially answering the open problem posed by Frank and Seiringer [4]. We note that for half-spaces this inequality has been recently obtained by Sloane [7].

1. Main result and discussion

For an open set $D \subset \mathbb{R}^n$ and $0 < \alpha < 2$ let

$$\mathcal{E}_D(u) = \frac{1}{2} \int_D \int_D \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}} \, dx \, dy, \quad u \in L^2(D).$$

The quadratic form $\mathcal{E}$ is, up to a multiplicative constant, the Dirichlet form of the censored stable process in D, see [1]. It has been shown [6] that for convex, connected open sets D and $1 < \alpha < 2$

$$\mathcal{E}_D(u) \geq \kappa_{n,\alpha} \int_D u^2(x) \, \delta_D^{-\alpha}(x) \, dx, \quad u \in C_c(D), \tag{1}$$

where $\delta_D(x) = \operatorname{dist}(x, D^c)$ and

$$\kappa_{n,\alpha} = \pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})} \frac{B\left(\frac{1+\alpha}{2}, \frac{2-\alpha}{2}\right) - 2^\alpha}{\alpha 2^\alpha} \tag{2}$$

is the *largest* constant for which (1) holds. In (2) B is the Euler beta function.

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On the other hand, if $0 < \alpha < n \wedge 2$, by Sobolev embedding [2, (2.3)], we have, e.g., for open convex, connected sets D

$$\mathcal{E}_D(u) + \|u\|_{L^2}^2 \geq c \left(\int_D |u(x)|^{2^*} \, dx \right)^{2/2^*}, \quad u \in C_c(D), \tag{3}$$

with some $c = c(D, \alpha) > 0$ and $2^* = 2n/(n - \alpha)$.

Comparing (1) and (3) an interesting question arises, whether the following Hardy-Sobolev-Maz'ya inequality

$$\mathcal{E}_D(u) \geq \kappa_{n,\alpha} \int_D u^2(x) \, \delta_D^{-\alpha}(x) \, dx + c \left(\int_D |u(x)|^{2^*} \, dx \right)^{2/2^*}, \quad u \in C_c(D), \tag{4}$$

holds for $1 < \alpha < n$ and convex domains D? A similar question was posed in [4, page 2].

The purpose of this note is to prove (4) for a half-space and balls, see Theorem 3 and Corollary 4. We would like to note that while writing this note a paper [7] of Sloane appeared, in which the author has proved (4) for half-spaces. However, our proof is different and we also obtain (4) for balls.

2. Proofs

We denote by $B_r = \{x \in \mathbb{R}^n : |x| < r\}$ the open Euclidean ball of radius $r > 0$, we set $B = B_1$ and by $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ we denote the $(n-1)$-dimensional unit sphere.

We define

$$L_D u(x) = \lim_{\varepsilon \to 0^+} \int_{D \cap \{|y - x| > \varepsilon\}} \frac{u(y) - u(x)}{|x - y|^{n+\alpha}} \, dy.$$

Note that L_D is, up to the multiplicative constant, the regional fractional Laplacian for an open set D, see [5].

Let

$$w_n(x) = (1 - |x|^2)^{\frac{\alpha - 1}{2}}, \quad x \in B \subset \mathbb{R}^n, n \geq 1. \tag{5}$$

We recall from [3, Lemma 2.1] that

$$-L_{(-1,1)} w_1(x) = \frac{(1 - x^2)^{\frac{-\alpha - 1}{2}}}{\alpha} \left(B\left(\frac{\alpha + 1}{2}, \frac{2 - \alpha}{2} \right) - (1 - x)^\alpha + (1 + x)^\alpha \right)$$

hence by [3, (2.3)]

$$-L_{(-1,1)} w_1(x) \geq c_1 (1 - x^2)^{\frac{-\alpha - 1}{2}} + c_2 (1 - x^2)^{\frac{-\alpha + 1}{2}}, \tag{6}$$

where

$$c_1 = \frac{B\left(\frac{\alpha+1}{2}, \frac{2-\alpha}{2}\right) - 2^\alpha}{\alpha}, \quad c_2 = \frac{2^\alpha - 2}{\alpha}.$$

Lemma 1. *We have for w_n defined in (5) and $n \geq 2$*

$$-L_B w_n(x) \geq \frac{c_1}{2} \int_{S^{n-1}} |h_n|^\alpha dh \cdot (1-|x|^2)^{-\frac{\alpha+1}{2}} + c_2 |S^{n-1}| \cdot (1-|x|^2)^{-\frac{\alpha-1}{2}}$$

Proof. Let $\mathbf{x} = (0, 0, \ldots, 0, x)$, $p = \frac{\alpha-1}{2}$. We have

$$\begin{aligned} -L_B w_n(\mathbf{x}) &= p.v. \int_B \frac{(1-|\mathbf{x}|^2)^p - (1-|y|^2)^p}{|\mathbf{x}-y|^{n+\alpha}}\, dy \\ &= \frac{1}{2} \int_{S^{n-1}} dh\ p.v. \int_{-xh_n - \sqrt{x^2h_n^2 - x^2 + 1}}^{-xh_n + \sqrt{x^2h_n^2 - x^2+1}} \frac{(1-|x|^2)^p - (1-|x+ht|^2)^p}{|t|^{1+\alpha}}\, dt. \end{aligned}$$

We calculate the inner principle value integral by changing the variable $t = -xh_n + u\sqrt{x^2h_n^2 - x^2 + 1}$

$$\begin{aligned} g(x,h) &:= p.v. \int_{-xh_n - \sqrt{x^2h_n^2 - x^2+1}}^{-xh_n + \sqrt{x^2h_n^2 - x^2+1}} \frac{(1-|x|^2)^p - (1-|x+ht|^2)^p}{|t|^{1+\alpha}}\, dt \\ &= p.v. \int_{-1}^{1} \frac{(1-x^2)^p - (1-u^2)^p(1-x^2+x^2h_n^2)^p}{|-xh_n + u\sqrt{x^2h_n^2 - x^2+1}|^{1+\alpha}} \sqrt{x^2h_n^2 - x^2+1}\, du \\ &= (1-x^2+x^2h_n^2)^{p-\alpha/2} p.v. \int_{-1}^{1} \frac{(1 - \frac{x^2h_n^2}{1-x^2+x^2h_n^2})^p - (1-u^2)^p}{|u - \frac{xh_n}{\sqrt{1-x^2+x^2h_n^2}}|^{1+\alpha}}\, du \\ &= (1-x^2+x^2h_n^2)^{-1/2} (-L_{(-1,1)} w_1)\left(\frac{xh_n}{\sqrt{1-x^2+x^2h_n^2}}\right). \end{aligned}$$

Hence by (6) we have

$$\begin{aligned} g(x,h) &\geq (1-x^2+x^2h_n^2)^{-1/2} \Bigg(c_1 \left(1 - \frac{x^2h_n^2}{1-x^2+x^2h_n^2}\right)^{\frac{\alpha-1}{2} - \alpha} \\ &\qquad + c_2 \left(1 - \frac{x^2h_n^2}{1-x^2+x^2h_n^2}\right)^{\frac{\alpha-1}{2} - \alpha + 1} \Bigg) \\ &= c_1 (1-x^2+x^2h_n^2)^{\alpha/2} (1-x^2)^{-\frac{\alpha+1}{2}} + c_2 (1-x^2+x^2h_n^2)^{\alpha/2-1} (1-x^2)^{-\frac{\alpha-1}{2}} \\ &\geq c_1 |h_n|^\alpha (1-x^2)^{-\frac{\alpha+1}{2}} + c_2 (1-x^2)^{-\frac{\alpha-1}{2}}. \end{aligned}$$

Thus

$$
\begin{aligned}
-L_B w_n(\mathbf{x}) &= \frac{1}{2}\int_{S^{n-1}} g(x,h)dh \\
&\geq \frac{c_1}{2}\int_{S^{n-1}} |h_n|^\alpha dh \cdot (1-x^2)^{-\frac{\alpha+1}{2}} + c_2|S^{n-1}|\cdot(1-x^2)^{-\frac{\alpha-1}{2}}
\end{aligned}
$$

and we are done. □

Corollary 2. *Let $1<\alpha<2$. Let w_n be as in (5) and let $n\geq 2$. Then for every $u\in C_c(B)$,*

$$
\begin{aligned}
\mathcal{E}_B(u) &:= \frac{1}{2}\int_B\int_B \frac{(u(x)-u(y))^2}{|x-y|^{n+\alpha}}\,dx\,dy \\
&\geq \frac{1}{2}\int_B\int_B \left(\frac{u(x)}{w(x)}-\frac{u(y)}{w(y)}\right)^2 \frac{w(x)w(y)}{|x-y|^{n+\alpha}}\,dx\,dy \\
&\quad + 2^\alpha\kappa_{n,\alpha}\int_B u^2(x)\,(1-|x|^2)^{-\alpha}dx + c_3\int_B u^2(x)\,(1-|x|^2)^{-\alpha+1}\,dx \qquad (7)
\end{aligned}
$$

Proof. The result follows from [3, Lemma 2.2] applied to $w=w_n$, Lemma 1, and the following formula [6, (7)]

$$
\kappa_{n,\alpha} = \kappa_{1,\alpha}\cdot\frac{1}{2}\int_{S^{n-1}} |h_n|^\alpha\,dh.
$$

□

Theorem 3. *Let $1<\alpha<2$ and $n\geq 2$. There exist a constant $c=c(\alpha,n)$ such that for every $0<r<\infty$ and $u\in C_c(B_r)$,*

$$
\begin{aligned}
\mathcal{E}_{B_r}(u) &:= \frac{1}{2}\int_{B_r}\int_{B_r} \frac{(u(x)-u(y))^2}{|x-y|^{n+\alpha}}\,dx\,dy \\
&\geq 2^\alpha\kappa_{n,\alpha}\int_{B_r} u^2(x)\,r^\alpha(r^2-|x|^2)^{-\alpha}dx + c\left(\int_{B_r}|u(x)|^{2^*}\,dx\right)^{2/2^*}, \qquad (8)
\end{aligned}
$$

where $2^*=2n/(n-\alpha)$.

Proof. By scaling, we may and do assume that $r=1$, that is we consider only the unit ball $B=B_1\subset\mathbb{R}^n$. Recall (5) and let $v=u/w_n$, $a_k=1-2^{-k}$ and $B_k=B(0,a_k)$. For $x,y\in B_{k_0}$ we have

$$
w_n(x)w_n(y) = w_1(|x|)w_1(|y|) \geq \sum_{k=k_0}^{\infty}(w_1^2(a_k)-w_1^2(a_{k+1})),
$$

thus for any $x, y \in B$

$$w_n(x)w_n(y) \geq \sum_{k=1}^{\infty} \left(\left(\frac{3}{2}\right)^{\alpha-1} - 1 \right) 2^{-k(\alpha-1)} 1_{B_k}(x) 1_{B_k}(y)$$
$$\geq \sum_{k=1}^{\infty} \frac{\alpha-1}{2} 2^{-k(\alpha-1)} 1_{B_k}(x) 1_{B_k}(y).$$

It follows that

$$\int_B \int_B \left(\frac{u(x)}{w_n(x)} - \frac{u(y)}{w_n(y)} \right)^2 \frac{w_n(x)w_n(y)}{|x-y|^{n+\alpha}} \, dx \, dy + \int_B v^2(x) w^2(x) \, dx$$

(9)

$$\geq \frac{\alpha-1}{2} \sum_{k=1}^{\infty} 2^{-k(\alpha-1)} \left(\int_{B_k} \int_{B_k} \frac{(v(x)-v(y))^2}{|x-y|^{n+\alpha}} \, dx \, dy + \int_{B_k} v^2(x) \, dx \right).$$

We write Sobolev inequality (3) for $D = B_k$ and a function v

(10)

$$\int_{B_k} \int_{B_k} \frac{(v(x)-v(y))^2}{|x-y|^{n+\alpha}} \, dx \, dy + \int_{B_k} v^2(x) \, dx \geq c \left(\int_{B_k} |v(x)|^{2^*} \, dx \right)^{2/2^*}.$$

The constant $c = c(\alpha, n)$ in (10) may be chosen such that it does not depend on k, because the radii a_k of B_k satisfy $1/2 \leq a_k \leq 1$. By (9) and (10) we obtain

$$\int_B \int_B \left(\frac{u(x)}{w_n(x)} - \frac{u(y)}{w_n(y)} \right)^2 \frac{w_n(x)w_n(y)}{|x-y|^{n+\alpha}} \, dx \, dy + \int_B v^2(x) \, dx$$
$$\geq c \sum_{k=1}^{\infty} 2^{-k(\alpha-1)} \left(\int_{B_k} |v(x)|^{2^*} \, dx \right)^{2/2^*}$$
$$\geq c \left(\sum_{k=1}^{\infty} \int_{B_k} 2^{\frac{-2^* k(\alpha-1)}{2}} |v(x)|^{2^*} \, dx \right)^{2/2^*}$$
$$\geq c' \left(\int_B w_n(x)^{2^*} |v(x)|^{2^*} \, dx \right)^{2/2^*} = c' \left(\int_B |u(x)|^{2^*} \, dx \right)^{2/2^*}.$$

By this and Corollary 2 we obtain (8). $\square$

Corollary 4. *Let $1 < \alpha < 2$, $n \geq 2$ and $\Pi = \mathbb{R}^{n-1} \times (0, \infty)$. There exist a constant $c = c(\alpha, n)$ such that for every $u \in C_c(B_r)$*

$$\mathcal{E}_\Pi(u) := \frac{1}{2} \int_\Pi \int_\Pi \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}} \, dx \, dy$$

$$\geq \kappa_{n,\alpha} \int_\Pi u^2(x) x_n^{-\alpha} \, dx + c \left(\int_\Pi |u(x)|^{2^*} \, dx \right)^{2/2^*}, \tag{11}$$

where $2^* = 2n/(n - \alpha)$.

Proof. By Theorem 3

$$\mathcal{E}_{B_r}(u) \geq \kappa_{n,\alpha} \int_{B_r} u^2(x) \delta_{B_r}(x)^{-\alpha} \, dx + c \left(\int_{B_r} |u(x)|^{2^*} \, dx \right)^{2/2^*},$$

where $\delta_{B_r}(x) = \operatorname{dist}(x, B_r^c)$. Let $x_r = (0, \ldots, 0, r) \in \Pi$, by translation and inequality $\delta_{B(x_r,r)}(x) \leq x_n$ we obtain

$$\mathcal{E}_{B(x_r,r)}(u) \geq \kappa_{n,\alpha} \int_{B(x_r,r)} u^2(x) x_n^{-\alpha} \, dx + c \left(\int_{B(x_r,r)} |u(x)|^{2^*} \, dx \right)^{2/2^*}.$$

The corollary follows by letting $r \to \infty$. □

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Faculty of Mathematics, University of Bielefeld, Postfach 10 01 31, D-33501 Bielefeld, Germany and Institute of Mathematics and Computer Science, Wrocław University of Technology, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

E-mail address: `bdyda (at) pwr wroc pl`